STEPWISE APPROXIMATION OF OPTIMUM CONTROLS

(STUPENCHATAIA APPROKSIMATSIIA OPTIMAL'NYKH UPRAVLENII)

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Iu.N. IVANOV
(Moscow)

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The problem of finding a simple optimum control law is considered. A control law which consists in changing the control position a given finite number of times is considered simple. The algorithm for selecting optimum control positions and optimum instants of switching from one position to the other is indicated. Examples are presented from the domain of optimum motion of a variable-mass body with constant thrust.

The realization of optimum laws for control functions is always associated with difficulties of technical nature, unless these laws are simple, for example in the case of constant or piecewise constant functions of time.

Below the problem of the best approximation to a complex control law by a simple law is considered, namely, the replacement of a complex continuous control function by a piecewise constant function with a given number of levels (steps). The control here has a prescribed number of optimum positions in place of an infinite number and these are shifted at optimum instants.

Such simple optimum control laws must be sought, for example, in problems of optimizing the power-limited motion of a variable-mass body (see [1], for example). It is known that in the absence of a constraint on the reactive thrust, the optimum law of its change represents a continuous function of time; the realization of such a law is difficult. On the other hand, if the condition of constant absolute value of the thrust along the trajectory (with optimum cut-off or without) is imposed in advance, then although such a law is simple, it gives a great loss in the functional, i.e. in payload. Such engine adjustment will be simple when the number of control positions is finite and prescribed, i.e. guarantees the engine a prescribed number of thrust levels. The problem of optimum selection of these levels and of the optimum instants of changing the levels arises.

Examples of solving particular problems of stepwise approximations of controls are known in the literature [2 and 3].

In [2] an optimum stepwise change in the weight of the power source is determined simply because of the special form of the functional of the problem and because of the upper bound on the derivative of the weight of the power source. A similar situation also holds in the case investigated in [3].

Let a dynamic system be described by the differential equations and boundary conditions

$$x_i^{\cdot} = f_i(x_j, u_k, t), \quad x_i(0) = x_i^{(0)}, \quad x_l(T) = x_l^{(1)} \qquad (i, j = 0, 1, \ldots, n \quad l = 1, \ldots, n;\ k = 1, \ldots, m) \tag{1}$$

Here the x_i are phase coordinates, u_k the control functions (the positions of the controls) and $x_0(T)$ is the control functional of the problem.

The solution of the variational problem on the extremum of the functional $x_0(T)$ yields the optimum controls $u_1^*(t), \ldots, u_m^*(t)$. Let us consider such a situation when the optimum law for one of the control functions, $u_1^*(t)$, say, is too complex for practical realization. Naturally, the problem arises of finding such a law $u_1(t)$, in place of $u_1^*(t)$, which would be simple to realize and at the same time would not "worsen" the value of the functional $x_0(T)$ by much.

Let us consider the control law to be simple if it consists of changing the positions of the controls a given finite number of times, i.e. the control function is piecewise-constant function. By using $N-1$ relay functions $\delta(t)$ which take the values 0 or 1, the piecewise-constant function, which takes on N values, may be represented as follows [4 and 5]:

$$u_1(t) = (\ldots(\alpha_1\delta_1 + \alpha_2)\delta_2 + \ldots + \alpha_{N-1})\delta_{N-1} + \alpha_N \tag{2}$$

Here the $\alpha_1, \ldots, \alpha_N$ are parameters defining the height of the steps. The N values of the control $u_1(t)$ are expessed thus

$$u_1^{(N)} = \alpha_N, \ldots, u_1^{(1)} = \alpha_1 + \ldots + \alpha_N \tag{3}$$

Let us introduce the operating time of the parameters $\alpha_1, \ldots, \alpha_N$.

The parameter α_N is connected throughout the process $(0 \leqslant t \leqslant T)$, the parameter α_{N-1} is connected when $\delta_{N-1} = 1$, the parameter α_2 is connected when $\delta_2\delta_3\ldots\delta_{N-1} = 1$, etc. The present times of operation of the parameters $t_{M1}, \ldots, t_{MN}$ are determined by differential expressions of the form

$$t_{Ms}^{\cdot} = \delta_s\delta_{s+1}\ldots\delta_{N-1} \qquad (s = 1, \ldots, N-1), \qquad t_{MN} = 1 \tag{4}$$

The total operating times are given by the integrals

$$T_{Ms} = \int_0^T \delta_s\delta_{s+1}\ldots\delta_{N-1}\,dt \qquad (s = 1, \ldots, N-1), \qquad T_{MN} = T \tag{5}$$

We subject the selection of the parameters $\alpha_1, \ldots, \alpha_N$ and the switching points of the relay functions $\delta_1, \ldots, \delta_{N-1}$ to the condition of an extremum of the functional $x_0(T)$. For this purpose we use the method of L.S.Pontriagin: we form the Hamiltonian H and we write the differential equations of the momenta

$$H = \sum_{i=0}^{n} p_i f_i [x_j, u_k, t, (\ldots(\alpha_1\delta_1 + \alpha_2)\delta_2 + \ldots + \alpha_{N-1})\delta_{N-1} + \alpha_N]$$
$$p_i^{\cdot} = -\partial H / \partial x_i \qquad (i, j = 0, 1, \ldots, n;\ k = 2, \ldots, m) \tag{6}$$

The optimum controls $u_2, \ldots, u_m$ are found by a standard method; to determine the optimum relay controls $\delta_1, \ldots, \delta_{N-1}$ the function H should be evaluated for the following sets of values of these controls at each time t

H_1	H_2	$\ldots$	H_{N-1}	H_N
$\delta_{N-1} = 0$	$\delta_{N-1} = 1$		$\delta_{N-1} = 1$	$\delta_{N-1} = 1$
	$\delta_{N-2} = 0$		$\ldots$	$\ldots$
			$\delta_2 = 1$	$\delta_2 = 1$
			$\delta_1 = 0$	$\delta_1 = 1$

The greatest (or least, depending on the nature of the extremum $x_0(T)$) value of H from the N evaluated quantities indicates the optimum set of values of the relay controls at the time t.

The following method may be used to find the optimum values of the parameters $\alpha_1,\ldots,\alpha_N$ (see [3 and 6], for example). Considering the parameters $\alpha_1,\ldots,\alpha_N$ to be phase coordinates, let us join the differential equations

$$\alpha_1^{\cdot} = 0, \ldots, \alpha_N^{\cdot} = 0$$

to the system (1).

The Hamiltonian function does not change here but N differential equations of the form

$$p_{\alpha,1}^{\cdot} = -\frac{\partial H}{\partial \alpha_1} = -\frac{\partial H}{\partial u_1}\delta_1\ldots\delta_{N-1}, \ldots, p_{\alpha,N} = -\frac{\partial H}{\partial u_1} \tag{7}$$

are joined to Equations (6) for the momenta.

The initial and final values of the phase coordinates are not fixed, hence the initial and final values of the momenta ~ are zero. Hence, the conditions for the selection of the optimum values of the parameters $\alpha_1,\ldots,\alpha_N$ follow

$$\int_0^T \frac{\partial H}{\partial u_1}\delta_1\ldots\delta_{N-1}\,dt = 0, \ldots, \int_0^T \frac{\partial H}{\partial u_1}\,dt = 0 \tag{8}$$

Using (4) and (5) the last formulas may be represented uniformly in the form

$$\int_0^{T_{Ms}} \frac{\partial H}{\partial u_1}\,dt_{Ms} = 0 \qquad (s = 1,\ldots,N-1), \qquad \int_0^T \frac{\partial H}{\partial u_1}\,dt = 0 \tag{9}$$

E x a m p l e s The criterion of optimum conditions for constant-thrust motion of a body of variable mass is the integral functional (see [1], say)

$$J = \int_0^T a^2 dt \qquad (a \text{ is the modulus of reactive acceleration})$$

If the motion consists of movement during the time T between two equilibrium points separated by a distance l in a forceless field, then the connection between the kinetic characteristics of the trajectory and the acceleration a is given by two differential equations anf boundary conditions

$$x^{\cdot} = v,\ v^{\cdot} = a\beta;\ x(0) = v(0) = v(T) = 0,\ x(T) = l$$

where $\beta = \pm 1$ is the thrust-vector direction.

Let us refer the present length x to the interval l, the present time t to the time of motion T, the velocity v to l/T, the acceleration a to l/T^2 and the functional J to l^2/T^3. Then the variational problem is written as

$$x^{\cdot} = v,\ v^{\cdot} = a\beta,\ J^{\cdot} = a^2;\ x(0) = v(0) = J(0) = v(1) = 0,\ x(1) = 1,\ \min J(1) \tag{10}$$

(here the notation for all the quantities is the same as before).

In the absence of a constraint on the control $a(t)$, the optimum laws of $a(t)$ and $\beta(t)$ have the form (the curve ∞ in Fig.1)

$$\begin{aligned} a &= 12(1/_2 - t), \quad \beta = 1 \quad (1/_2 \geqslant t \geqslant 0) \\ a &= 12(t - 1/_2), \beta = -1 \quad (1 \geqslant t \geqslant 1/_2), \end{aligned} \qquad J(1) = 12 \tag{11}$$

Presented below are results of computations of certain stepwise optimum laws $a(t)$ and the use of the proposed method is shown in the last of them.

1) For $a = \alpha_1$ the optimum laws $a(t)$, $\beta(t)$ are (Fig.1, curve 1)

$$a = 4, \quad \beta = 1 \quad (^1/_2 \geqslant t \geqslant 0), \qquad J(1) = 16 \qquad (12)$$
$$a = 4, \quad \beta = -1 \quad (1 \geqslant t \geqslant {}^1/_2),$$

2) For $a = \alpha_1\delta_1$ the optimum laws $\alpha(t)$, $\beta(t)$ are (Fig.1, curve 2)

$$a = 4.5, \quad \beta = 1 \quad (^1/_3 \geqslant t \geqslant 0),$$
$$a = 0 \quad (^2/_3 \geqslant t \geqslant {}^1/_3), \qquad J(1) = 13.5 \qquad (13)$$
$$a = 4.5, \quad \beta = -1 \quad (1 \geqslant t \geqslant {}^2/_3),$$

3) For $a = \alpha_1\beta_1 + \alpha_2$ the optimum laws $a(t)$, $\beta(t)$ are (Fig.1, curve 3)

$$a = 4.8, \quad \beta = 1 \quad (^1/_4 \geqslant t \geqslant 0)$$
$$a = 1.6, \quad \beta = 1 \quad (^1/_2 \geqslant t \geqslant {}^1/_4) \qquad J(1) = 12.8 \qquad (14)$$
$$a = 1.6, \quad \beta = -1 \quad (^3/_4 \geqslant t \geqslant {}^1/_2)$$
$$a = 4.8, \quad \beta = -1 \quad (1 \geqslant t \geqslant {}^3/_4)$$

4) For $a = (\alpha_1\delta_1 + \alpha_2)\delta_2$ the optimum laws $a(t)$, $\beta(t)$ are (Fig.1, curve 4)

$$a = 5, \quad \beta = 1 \quad (^1/_5 \geqslant t \geqslant 0)$$
$$a = 2.5, \quad \beta = 1 \quad (^2/_5 \geqslant t \geqslant {}^1/_5)$$
$$a = 0 \quad (^3/_5 \geqslant t \geqslant {}^2/_5) \qquad J(1) = 12.5 \qquad (15)$$
$$a = 2.5, \quad \beta = -1 \quad (^4/_5 \geqslant t \geqslant {}^3/_5)$$
$$a = 5, \quad \beta = -1 \quad (1 \geqslant t \geqslant {}^4/_5)$$

Levels with given (null) amplitude are included in the composition of the controls in the examples 2 and 4; only their optimum position is indicated for sections with a given magnitude control.

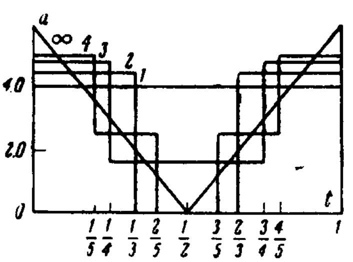

Fig. 1

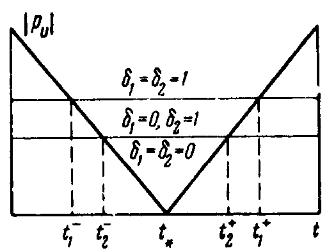

Fig. 2

For the three-step control function

$$a = (\alpha_1\delta_1 + \alpha_2)\,\delta_2 \qquad (16)$$

with a zero level α_3 (see example (4)), the differential equations of the phase coordinates, the Hamiltonian function, the differential equations of the momenta and the equations for the selection of the optimum parameters α_1, α_2 are according to (10), (6) and (8)

$$x^{\cdot} = v, \qquad v^{\cdot} = \beta\,(\alpha_1\delta_1 + \alpha_2)\,\delta_2, \qquad J^{\cdot} = \alpha_1^2\delta_1\delta_2 + 2\alpha_1\alpha_2\delta_1\delta_2 + \alpha_2^2\delta_2, \qquad p_v^{\cdot} = -p_x$$

$$H = -\alpha_1^2\delta_1\delta_2 - 2\alpha_1\alpha_2\delta_1\delta_2 - \alpha_2^2\delta_2 + p_v\beta\,(\alpha_1\delta_1 + \alpha_2)\,\delta_2 + p_x v, \qquad p_x^{\cdot} = 0 \qquad (17)$$

$$\alpha_1\int_0^1 \delta_1\delta_2\,dt + \alpha_2\int_0^1 \delta_1\delta_2\,dt = \frac{1}{2}\int_0^1 p_v\beta\delta_1\delta_2\,dt,\ \alpha_1\int_0^1 \delta_1\delta_2\,dt + \alpha_2\int_0^1 \delta_2\,dt = \frac{1}{2}\int_0^1 p_v\beta\delta_2\,dt \quad (17)\ \text{cont.}$$

The solution for the differential equation for the momentum p_v may be represented as

$$p_v = c\,(t_* - t) \quad (18)$$

The optimum controls $\beta(t)$, $\delta_1(t)$ and $\delta_2(t)$,which creates a maximum for the function H are subject to the restrictions

$$\beta = \operatorname{sign} p_v(t) \qquad (\beta p_v = |p_v|) \quad (19)$$

$$\delta_1 = 1 \quad \text{for } \Delta_1 > 0, \qquad \delta_1 = 0 \quad \text{for } \Delta_1 < 0 \quad (\Delta_1 = \alpha_1(-\alpha_1 - 2\alpha_2 + |p_v|)) \quad (20)$$

$$\delta_2 = 1 \quad \text{for } \Delta_2 > 0, \qquad \delta_2 = 0 \quad \text{for } \Delta_2 < 0 \quad (\Delta_2 = \alpha_2(|p_v| - \alpha_2) + \delta_1\Delta_1) \quad (21)$$

The parameter α_2 may only be positive since $a > 0$ (see (16)). The parameter α_1 may be positive or negative, in the latter case $|\alpha_1| < \alpha_2$, since $a > 0$. Let us first consider the case $\alpha_1 > 0$.

Let $-\alpha_1 - \alpha_2 + |p_v| - \alpha_2 > 0$, then $\Delta_1 > 0$ for $\alpha_1 > 0$ (see (20)) and $\delta_1 = 1$. Hence, the expression $|p_v| - \alpha_2$ is known to be positive; therefore, $\Delta_2 > 0$ (see (21)) and $\delta_2 = 1$. Let $|p_v| - a_2 < 0$, then it is known that $\Delta_1 < 0$, and, therefore, $\delta_1 = 0$. Hence, $\Delta_2 < 0$ and $\delta_2 = 0$.

These reasonings lead to the deduction: if $\delta_1 = 1$, then it is known that $\delta_2 = 1$; if $\delta_2 = 0$, it is then known that $\delta_1 = 0$, i.e. the section $\delta_2 = 0$ is located within the section $\delta_1 = 0$, and the section $\delta_1 = 1$ within the section $\delta_2 = 1$. The disposition of the sections is shown in Fig.2 for $|p_v(t)|$ the piecewise-linear function (18). Here $t_1^-, t_2^-, t_2^+, t_1^+$ are roots of Equations

$$\Delta_1(t_1^-) = -\alpha_1 - 2\alpha_2 + |c|(-t_1^- + t_*) = 0$$
$$\Delta_1(t_1^+) = -\alpha_1 - 2\alpha_2 + |c|(t_1^+ - t_*) = 0 \quad (22)$$

$$\Delta_2(t_2^-) = |c|(-t_2^- + t_*) - \alpha_2 = 0, \qquad \Delta_2(t_2^+) = |c|(t_2^+ - t_*) - \alpha_2 = 0 \quad (23)$$

Hence, in particular, there follows

$$t_1^+ + t_1^- = 2t_*, \qquad t_2^+ + t_2^- = 2t_* \quad (24)$$

The optimum controls $\beta(t)$, $\delta_1(t)$ and $\delta_2(t)$ may be written with the aid of parameters t_1^-, t_2^-, t_*, t_2^+ and t_1^+ as follows:

$$\begin{array}{llll} \delta_1 = 1, & \delta_2 = 1, & \beta = 1 & (t_1^- \geqslant t \geqslant 0) \\ \delta_1 = 0, & \delta_2 = 1, & \beta = 1 & (t_2^- \geqslant t \geqslant t_1^-) \\ \delta_1 = 0, & \delta_2 = 0, & & (t_2^+ \geqslant t \geqslant t_2^-) \\ \delta_1 = 0, & \delta_2 = 1, & \beta = -1 & (t_1^+ \geqslant t \geqslant t_2^+) \\ \delta_1 = 1, & \delta_2 = 1, & \beta = -1 & (1 \geqslant t \geqslant t_1^+) \end{array} \quad (25)$$

After integration of the system of equations of motion with the boundary conditions $x(0) = v(0) = v(1) = 0$, $x(1) = 1$ we find the following constraints:

$$2t_* = 1, \quad \alpha_1 t_1^-(1 - t_1^-) + \alpha_2 t_2^-(1 - t_2^-) = 1 \quad (26)$$

The parameters α_1, α_2 are expressed in terms of $|c|, t_1^-, t_2^-$, thus

$$\alpha_1 = {}^1/_4|c|\,t_2^-, \qquad \alpha_2 = {}^1/_4|c|\,(1 - t_2^- - t_1^-) \quad (27)$$

We determine $t_1^- = {}^1/_5$, $t_2^- = {}^2/_5$ from Equations (22) and (23), we find $|c| = 25$ from the second equation of (26) and, finally we find $\alpha_1 = 2.5$ and $\alpha_2 = 2.5$.

The form of the optimum controls is presented above in (15).

If the parameter α_1 is considered negative, then as compared with the case considered, the optimum law $\delta_1(t)$ changes ($\delta_1 = 0$ for $1/5 \geqslant t \geqslant 0$ and for $1 \geqslant t \geqslant 4/5$, $\delta_1 = 1$ for $2/5 \geqslant t \geqslant 1/5$ and for $4/5 \geqslant t \geqslant 3/5$; δ_1 is not defined for $3/5 \geqslant t \geqslant 2/5$) as do the parameters α_1, α_2 ($\alpha_1 = -2.5, \alpha_2 = 5$). It is interesting to note that the optimum laws $\alpha(t)$, $\beta(t)$, as well as the magnitude of the functional J remain unchanged here.

In conclusion, let us make a few general remarks.

1. The ambiguity of the representation of the step control function in terms of the parameters $\alpha_1, \ldots, \alpha_N$ and the relay controls $\delta_1, \ldots, \delta_{N-1}$ may be detected by using the following discussion (let us do this for $N = 2$).

Let us assume that an optimum two-step control has been constructed

$$u_1 = \alpha_1\delta_1 + \alpha_2 \tag{28}$$

i.e. the relay function δ_1 and the parameters α_1 and α_2 have been chosen. Let us replace the function δ_1 by $\delta_1' = 1 - \delta_1$ and let us find the parameters α_1' and α_2', composing the control u_1'

$$u_1' = \alpha_1'\delta_1' + \alpha_2' \tag{29}$$

such that $u_1'(t) \equiv u_1(t)$. For $\delta_1 = 0$ we have $u_1 = \alpha_1$, $\delta_1' = 1$, and $u_1' = \delta_1' + \alpha_2'$; for $\delta_1 = 1$ we have $u_1 = \alpha_1 + \alpha_2$, $\delta_1' = 0$, and $u_1' = \alpha_2'$. Therefore, for the identity $u_1'(t) \equiv u_1(t)$ compliance with the conditions

$$\alpha_2 = \alpha_1' + \alpha_2', \quad \alpha_1 + \alpha_2 = \alpha_2' \qquad \text{or} \qquad \alpha_1' = -\alpha_1, \quad \alpha_2' = \alpha_2 + \alpha_2 \tag{30}$$

is necessary.

Hence, the second representation of the control function has been obtained which does not agree with the first but which yields the same law $u_1(t)$ and, therefore, the same magnitude of the checking functional.

2. It was indicated in the initial formulation of the problem considered that all the parameters $\alpha_1, \ldots, \alpha_N$ have been selected from optimum considerations. If the control function is constrained by the limits $1 \geqslant u_1 \geqslant 0$, then the constraints

$$\max\,[(\ldots(\alpha_1\delta_1 + \alpha_2)\,\delta_2 + \ldots + \alpha_{N-1})\,\delta_{N-1} + \alpha_N] \leqslant 1$$
$$\min\,[(\ldots(\alpha_1\delta_1 + \alpha_2)\,\delta_2 + \ldots + \alpha_{N-1})\,\delta_{N-1} + \alpha_N] \geqslant 0$$

are imposed on the parameters $\alpha_1, \ldots, \alpha_N$.

In particular, the boundary may be included in the composition of the optimum step control, as has been done in the examples 2 and 4. Let us present an example of writing a three-step control $u_1(t)$ which includes the lower 0 and the upper 1 boundaries

$$u_1 = ((1 - \alpha_2)\,\delta_1 + \alpha_2)\,\delta_2$$

Here we must have $1 \geqslant \alpha_2 \geqslant 0$. It is assumed that in the optimum case the control u_1 also takes on intermediate values.

3. Apparently the step control function approximates the continuous control function "better" (in the sense of the checking functional) as the number N increases. If $x_0{}^*(T)$ denotes the optimum value of the functional for a continuous control u_1^* and $x^{(N)}(T)$ denotes the optimum value of the functional for the step control $u_1^{(N)}$, then

$$|\,x_0{}^{(N)}(T) - x_0{}^*(T)\,| \to 0 \qquad \text{for } N \to \infty$$

4. A numerical approach to the solution of the problem of the stepwise approximation of controls without using the representation (2) with relay functions can be mentioned. Let us take the desired N control levels $u_1(t)$ $(u_1^{(1)}, \ldots, u_1^{(N)})$ and solve the problem by using the maximum principle.

The times to change levels are determined from the condition of the extremum of the Hamiltonian function and the optimum amplitudes of the levels from the condition of the extremum of the functional of the problem. This latter procedure requires implication of a numerical method of steepest descent type. The method with relay functions yields analytical expressions for the selection of the optimum amplitudes of the levels.

5. If the original variational problem (1) with the stepwise control is not subject to analytic solution, then the question arises of the selection of the numerical method for solving the boundary value problem. In addition to satisfying the boundary conditions on the phase coordinates, the method described above requires satisfaction of the conditions for selecting the optimum parameters which are representable either as the integrals (8) or as the differential equations (7) with zero boundary conditions for the momenta. One of the possible methods of solving this boundary value problem is the reduction to a Cauchy problem and the selection of deficient initial conditions constrained by some method of solving algebraic equations, e.g. the Newton method.

Let us note that in this case the numerical approach mentioned in the remark 4 may be used with equal success.

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